Lecture 24

Plan

- §9.1, 9.3
  finish recalling linear algebra

- Part of §9.4
  Introdue the defn of linear (differential) system.

§9.1, 9.3

Algebraic operation of matrices

- Scalar multiplication.

Let $r \in \mathbb{R}$ and $A = [a_{ij}]$ be a matrix.

("$A = [a_{ij}]$" means "$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$")

Then $rA = [ra_{ij}]$.

$$-A = (-1)A = [-a_{ij}].$$

E.g.: $2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\times1 \\ 2\times2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$$3\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 6 & 3 \end{bmatrix}$$

$$-\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix}$$

Matrix Addition.

If both A and B have the same # of columns and rows, then we can add them up or take their difference:

If $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$

then $A + B = [a_{ij} + b_{ij}]_{m\times n}$

$A - B = [a_{ij} - b_{ij}]_{m\times n}$

E.g

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1+2 & 0+3 \\ 1+4 & 2+5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 \\ 5 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & 2-0, & 3-0 \\ 4-0 & 5-1 & 6-0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 \\ 4 & 4 & 6 \end{bmatrix}$$

Warning: If two matrices A, B have different numbers of rows or columns, then we cannot add them up or take their difference.

- Matrix multiplication

Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{n\times p}$

Then we can take their product

$$C = AB$$

C will be an $m\times p$ matrix,

$$C = [c_{ij}]_{m\times p}$$

Where c_{ij} = ith row of A × jth column of B

$$= [a_{i1}, \cdots, a_{in}] \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

$$= \sum_{k=1}^{n} a_{ik} b_{kj}$$

E.g:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \text{1st row of A} \times \text{1st column of B} \\ \text{2nd row of A} \times \text{1st column of B} \\ \text{3rd row of A} \times \text{1st column of B} \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 0 + 2\cdot 1 + 3\cdot 2 \\ 4\cdot 0 + 5\cdot 1 + 6\cdot 2 \\ 7\cdot 0 + 8\cdot 1 + 9\cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 17 \\ 26 \end{bmatrix}$$

E.g

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1\cdot 0 + 2\cdot 0 & 1\cdot 0 + 2\cdot 5 \\ 3\cdot 0 + 4\cdot 0 & 3\cdot 0 + 4\cdot 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 10 \\ 0 & 20 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 15 & 20 \end{bmatrix}$$

Warning: In general, $AB \neq BA$

Useful formulars:

Let A, B, C be matrices and $k \in \mathbb{R}$.

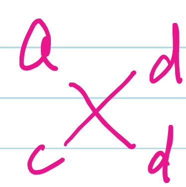
Then the following holds providing they are well defined:

- $A + B = B + A$
- $k(A+B) = kA + kB$
- $(AB)C = A(BC)$
- $A(B+C) = AB + AC$.

- Determinant for $n \times n$ matrices

(for 2×2 matrix)

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the determinant of $A = ad - bc$.



Notation: $\det(A)$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

(for 3×3 matrix)

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

the determinant of A is defined as

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Eg: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1\times 4 - 2\times 3 = -2$

- Inverse.

Let A be an $n\times n$ matrix. We say an $n\times n$ matrix B is the inverse of A if

$$AB = BA = I_n$$

("I_n" denotes the $n\times n$ identity matrix $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$)

E.g:

$$\underbrace{\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I_2}$$

$$\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I_2}$$

$\Rightarrow$ B is the inverse of A.

(A is the inverse of B)

E.g.:

$$\underbrace{\begin{bmatrix} \frac{3}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I_3}$$

You can verify $BA = I_3$

Hence B is the inverse of A.

(A is the inverse of B)

Thm: If $AB = I_n$, then we must have $BA = I_n$.

Def^n: We say an $n \times n$ matrix A is invertible if A has an inverse

Thm: A is invertible $\Leftrightarrow \det(A) \neq 0$.

§9.4 Linear (differential) system

Defn A matrix-valued function or simply matrix function is a matrix whose entries are functions

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & & & \vdots \\ a_{m1}(t) & a_{m2}(t) & \cdots & a_{mn}(t) \end{bmatrix}$$

We say $A(t)$ is differentiable if every of its entries is differentiable. In this case, The derivative of A is defined as

$$A'(t) = \frac{dA}{dt} = \left[\frac{da_{ij}}{dt}\right]_{m\times n}$$

E.g.:

$$A(t) = \begin{bmatrix} \sin t & 1 \\ t & \cos t \end{bmatrix}$$

$$\Rightarrow \quad A'(t) = \frac{dA}{dt} = \begin{bmatrix} \cos t & 0 \\ 1 & -\sin t \end{bmatrix}$$

Eg

$$\vec{x}(t) = \begin{bmatrix} 1 \\ t^2 \\ e^t \end{bmatrix}$$

$\vec{x}$: Vector function

x: scalar function

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 \\ 2t \\ -e^{-t} \end{bmatrix}$$

Differentiation Rules for matrix functions

- $\frac{d}{dt}(A+B) = \frac{dA}{dt} + \frac{dB}{dt}$
- $\frac{d}{dt}(kA) = k\frac{dA}{dt}$ ($k \in \mathbb{R}$ is constant)
- $\frac{d}{dt}(AB) = A\frac{dB}{dt} + \frac{dA}{dt}B$

1st-order linear (differential) system

A system of n 1st-order linear D.E

a collection of equations

takes the following form:

$$\begin{cases} x_1' = P_{11}(t)x_1 + P_{12}(t)x_2 + \cdots + P_{1n}(t)x_n + f_1(t) \\ x_2' = P_{21}(t)x_1 + P_{22}(t)x_2 + \cdots + P_{2n}(t)x_n + f_2(t) \\ \vdots \\ x_n' = P_{n1}(t)x_1 + P_{n2}(t)x_2 + \cdots + P_{nn}(t)x_n + f_n(t) \end{cases}$$

We normally prefer to write the above in the matrix form:

Write

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \Rightarrow \frac{d\vec{x}}{dt} = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$$

Write

$$P(t) = [p_{ij}(t)]_{1 \le i,j \le n}$$

$$\vec{f}(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

Then the above system of linear D.Es can be written as

matrix equation

$$\frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{f}(t) \qquad (*)$$

$$\left(\begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix} = \begin{bmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix} \right)$$

- $(*)$ is called the normal form of D.E system
- If $\vec{f}(t) \neq 0$, then $(*)$ is called a nonhomogeneous system

And "$\frac{d\vec{x}}{dt} = P(t)\vec{x}$" is called the associated homogeneous system.

E.g: Consider the first order linear differential system:

Note: $x = x(t)$, $y = y(t)$

$$\begin{cases} x' = 4x - 3y \\ y' = 6x - 7y \end{cases} \quad (1)$$

Q1: write it as a matrix equation

Note (1) $\Longleftrightarrow$

$$\underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\frac{d\vec{x}}{dt}} = \underbrace{\begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}}$$

write $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $P = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}$

$$\Rightarrow \frac{d\vec{x}}{dt} = P\vec{x}.$$

Q2: Verify $\vec{x}_1(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$ is a soln

to the system

Way 1: Note $\begin{cases} x = 3e^{2t} \\ y = 2e^{2t} \end{cases}$. We need to

verify they satisfy the two eqns:

$$\begin{cases} x' = 4x - 3y & (1) \\ y' = 6x - 7y & (2) \end{cases}$$

For (1), $x' = (3e^{2t})' = 6e^{2t}$

$$4x - 3y = 4 \cdot 3 \cdot e^{2t} - 3 \cdot 2e^{2t}$$

$$= 6e^{2t}$$

$$\Rightarrow x' = 4x - 3y$$

For (2), E.X.

↙ Recommended

Way 2: Recall the matrix eqn: $\frac{d\vec{x}}{dt} = P\vec{x}$

We need to verify $\frac{d\vec{x}_1}{dt} = P\vec{x}_1$

$$\vec{x}_1 = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} \Rightarrow \frac{d\vec{x}_1}{dt} = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix}$$

$$P\vec{x}_1 = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} 4 \cdot 3 \cdot e^{2t} - 3 \cdot 2e^{2t} \\ 6 \cdot 3 \cdot e^{2t} - 7 \cdot 2e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix}$$

$$\Rightarrow \frac{d\vec{x}_1}{dt} = P\vec{x}_1$$

Q3: Verify $\vec{x}_2(t) = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$ is also a solution to the system

E.x: Use Way 2 (imitate Q2)

Q4: Given any constants c_1, c_2, verify $c_1\vec{x}_1 + c_2\vec{x}_2$ is a solution to the system.

Here $\vec{x}_1 = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$ as in Q2

$\vec{x}_2 = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$ as in Q3.

A: We need to verify $\frac{d\vec{x}}{dt} = P\vec{x}$

is satisfied by $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$

Let $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$.

Then $\vec{x} = c_1 \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$

$$= \begin{bmatrix} 3c_1 e^{2t} + c_2 e^{-5t} \\ 2c_1 e^{2t} + 3c_2 e^{-5t} \end{bmatrix}$$

$$\Rightarrow \frac{d\vec{x}}{dt} = \begin{bmatrix} 6c_1 e^{2t} - 5c_2 e^{-5t} \\ 4c_1 e^{2t} - 15c_2 e^{-5t} \end{bmatrix}$$

$$P\vec{x} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} 3c_1 e^{2t} + c_2 e^{-5t} \\ 2c_1 e^{2t} + 3c_2 e^{-5t} \end{bmatrix}$$

$= \cdots \leftarrow$ Ex.

E.X: Then check $\frac{d\vec{x}}{dt} = P\vec{x}$.